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Relativistic Gamow Vectors

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Whereas in Dirac quantum mechanics and relativistic quantum field theory one uses Schwartz space distributions, the extensions of the Hilbert space that we propose uses Hardy spaces. The in- and out-Lippmann–Schwinger kets of scattering theory are functionals in two rigged Hilbert space extensions of the same Hilbert space. This hypothesis also allows to introduce generalized vectors corresponding to unstable states, the Gamow kets. Here the relativistic formulation of the theory of unstable states is presented. It is shown that the relativistic Gamow vectors of the unstable states, defined by a resonance pole of the *S*- matrix, are classified according to the irreducible representations of the semigroup of the Poincaré transformations (into the forward light cone). As an application the problem of the mass definition of the intermediate vector boson *Z* is discussed and it is argued that only one mass definition leads to the exponential decay law, and that is not the standard definition of the on-the-mass-shell renormalization scheme.

KEY WORDS: relativistic Gamow vectors; Hilbert space.

1. INTRODUCTION

In nature we observe stable systems like atoms, molecules, electrons and protons, and also unstable systems, like excited states, unstable nuclei, resonances, etc. If we consider elementary particles then all but electrons, protons, and photons are unstable. From the practical point of view we thus see that the description of the unstable states is very important.

Existence of unstable states and their inclusion in quantum theory causes problems. It is clear that unstable states cannot be present in the asymptotic initial and final states because their lifetime is finite and they should be included only as the intermediate states in the interaction, as the propagators between the points of creation and annihilation. One of the fundamental properties of quantum mechanics is the unitarity of the *S* matrix. Any submatrix of the *S* matrix is not unitary. Physically this means that if some states are artificially excluded from the theory then the matrix *S* corresponding to the remaining set of the states will not be unitary. We thus see that the exclusion of the unstable states from the

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asymptotic initial and final states can lead to breaking of the S matrix unitarity which might eventually lead to the rejection of the probabilistic interpretation of quantum mechanics. In the classical paper, Veltman (1964) considers a field theoretic model with two scalar particles A and φ with the corresponding masses M and $m(M > 2m)$ and the interaction term $\sim A\varphi^2$. In such a theory the particle A is unstable and decays into two φ particles. The particle A, as the unstable particle, cannot appear in the asymptotic states. Veltman demonstrates that because of the condition $M > 2m$ the perturbation series becomes divergent at any value of the coupling constant so the theory in such a form is incapable to generate consistent results. The solution of this difficulty is the derivation of the perturbation series in the kinematic region where the perturbation series can be consistently derived and then obtain the results in the physical region by the analytic continuation. This is equivalent to the modification of the propagator for particle A. It turns out that the theory where in the asymptotic states there appear only particle φ and in the intermediate states there are particles φ and A with the modified propagators is unitary, renormalizable, and causal so it fulfills all the fundamental requirements. We thus see that the unstable states can be excluded from the asymptotic states but their propagators have to be modified and the theory remains consistent.

In practical calculations the Breit–Wigner formula is used for the propagator for the unstable states

$$\frac{\frac{1}{2}\Gamma}{E - E_R - \frac{i}{2}\Gamma} \tag{1}$$

where E_R and Γ are the energy and width of the resonance and E is the energy of the process.

Most practical calculations for the resonances are carried out using Eq. (1) or its relativistic generalization. Equation (1) implicitly contains a series of important properties of resonances which can but do not have to be confirmed by experiment, like, e.g., short-time behavior of the unstable states from the instant of production or the long-time behavior of the decay process.

The fact that unstable states are not included as asymptotic states poses a question if one needs to introduce the quantum mechanical state vector that corresponds to the resonance. The answer to this question can only be found analyzing the theory with the unstable states. In nature there are unstable states, like unstable nuclei with the mean lifetime equal to thousands of years. On the other extreme the lifetime of the hadronic resonances is of the order 10^{-24} s. The lifetime of the free neutron is approximately 887 s. and it is the isospin partner of the stable proton. Excluding from the theory the unstable states altogether would mean the rejection of such a useful notion like the isospin. It is thus clear that the answer to our question has been answered long ago, at the time when the isospin was introduced. The state of the neutron has been defined in exactly the same way as the state of the proton and the stable proton and unstable neutron were considered different states of the same physical system. In such a description the important fact that the neutron decays has been completely neglected. This description in some way was forced on by quantum mechanics in the Hilbert space with the unitary and reversible evolution while the the time evolution of the unstable states is irreversible. In this way we come to the very problem of the description of the unstable states in quantum mechanics which can be stated that the consistent description of the unstable states in the *Hilbert space quantum mechanics* is not possible and we must extend or generalize the conventional quantum mechanics. One of the possibilities is the introduction of non-hermitian Hamiltonians. In such models the probability is not conserved and also the time evolution operator of a free unstable system is nonunitary. This would lead to the modification of the probabilistic interpretation of quantum mechanics.

The space of states of conventional quantum mechanics is the Hilbert space. It is already not suffcient in the conventional quantum mechanics since the eigenstates of the momentum operator (plane waves) do not belong to the Hilbert space. One can avoid this difficulty through the systematic use of the normalizable wave packets but in practical calculations one mostly uses plane waves and obtains correct results without the computational complications of the wave packets. The justification of the plane waves method and of the Dirac formalism was given by Gel'fand and Vilenkin (1964) by the extension of Hilbert space to the triplet of spaces (rigged Hilbert space)

$$\Phi \subset \mathcal{H} \subset \Phi^{\times}, \tag{2}$$

where $\mathcal{H}$ is the conventional Hilbert space of the system, Φ is the dense subspace of the Hilbert space $\mathcal{H}$ with stronger topology and $\Phi^{\times}$ is the space dual to the space Φ (the space of the antilinear functionals of the space Φ).[2] The plane waves belong to the space $\Phi^{\times}$ and as the generalized eigenstates of the momentum operator they can be used without any mathematical inconsistencies.

The operators in the Hilbert space can be extended to the whole rigged Hilbert space. The operators in the rigged Hilbert space have properties which are not the same as in the Hilbert space. One of such properties is that the generalized eigenvalues in the rigged Hilbert space of the hermitian operator in the Hilbert space are not necessarily real.

Experimentally, the main effect of resonances is the bump at certain energy values in the differential cross-section for a given partial wave. Theoretically, the resonances are introduced as the poles on the unphysical sheet at a complex energy value of the S matrix for a given partial wave. It can be shown that this theoretical description is compatible with all experimental signatures of a resonance. The existence of the pole of the S matrix is an important mathematical property and it permits a precise mathematical description of a resonance in the scattering

[2] Notice that the space dual to the Hilbert space is identical to this Hilbert space (Riesz theorem).

experiments. This analysis leads to the following conclusions for the energies in the neighborhood of the S matrix pole:

1. S matrix element is the sum of the resonance contributions and a slowly varying background.
2. The vectors corresponding to the initial and final asymptotic states form two different rigged Hilbert spaces.
3. One can define the vector in the rigged Hilbert space that corresponds to an unstable state. This vector is derived from the Cauchy integral around the S matrix pole; it is called Gamow vector.
4. The two rigged Hilbert spaces of the initial and final states lead in a natural way to a mathematical differentiation between the prepared in-state vectors and the observed out-state vectors. The space of energy wave functions for the in-states will be Hardy spaces of analytic functions on the lower half plane and the space of energy wave functions for the out-states will be Hardy spaces of analytic functions on the upper half plane of the second sheet of the S matrix.

   As a consequence of the different Hardy space property for states and observables one obtains the time asymmetry for the Born probabilities of an observed out-state in a prepared in-state. This is a manifestation of causality, some call it microphysical irreversibility.

We will discuss these properties in the remaining part of the paper. Let us mention here that the formalism is fully relativistic, which manifests itself in the fact that the vectors of an unstable state form the semigroup representation of the Poincaré group transformations into the forward light cone. From the mathematical properties of the rigged Hilbert spaces it follows that the evolution of such vectors is allowed only for into the forward light cone in particular for time translations only for $t \geq 0$.

The vectors of the unstable states are an explicit realization of Veltman's idea, they lead to the *exact* exponential decay and they also exactly fulfill the Fermi's golden rule. They also lead the *exact* relation between the lifetime and the resonance width $\tau = \hbar/\Gamma$. The stable states can be obtained as the limit $\Gamma \to 0$. One can thus see that the formalism of the rigged Hilbert space allows to formulate the unified theory of the stable and unstable states with physically expected properties.

2. RIGGED HILBERT SPACE

2.1. General Discussion

The eigenvectors of the momentum operator in quantum mechanics are the solutions of the equation

$$\hat{p}\psi_p = p\psi_p. \tag{3}$$

In the position representation $\hat{p} = (\hbar/i)(\partial/\partial x)$ and ψ_p is equal

$$\psi_p(x) = e^{ipx}. \tag{4}$$

The function $\psi_p(x)$ has an infinite norm, because $|\psi_p(x)|^2 = 1$ and the integral $\int |\psi_p(x)|^2 \, d^3x$ is divergent. The eigenvectors of the $\hat{p}$ operator do not belong to the Hilbert space L^2.

In practical calculations the function $\psi_p(x)$ is frequently used as the eigenvector of the operator $\hat{p}$, in spite of the fact that it does not belong to the Hilbert space and its use always leads to the correct results. The justification of such a procedure is given in (Antoine, 1969; Bohm, 1966, 1978a; Gel'fand and Vilenkin, 1964; Roberts, 1966). Let $\mathcal{H}$ be the Hilbert space of our physical system. In this space the position operator $\hat{x}$ and momentum operator $\hat{p}$ are unbounded and their eigenstates do not belong to $\mathcal{H}$. To define the rigged Hilbert space one first introduces the dense subspace $\Phi \subset \mathcal{H}$ with stronger topology defined by the countable number of norms

$$\Phi \ni \varphi_n \underset{n\to\infty}{\longrightarrow} \varphi \equiv \|A^p(\varphi_n - \varphi)\| \underset{n\to\infty}{\longrightarrow} 0, \qquad p = 0, 1, 2 \ldots \tag{5}$$

where $A = H - 1/2$ and H is the Hamiltonian of the system. The operators $\hat{x}$ and $\hat{p}$ are continuous in the space Φ. One then considers the space $\Phi^\times$ dual to Φ, i.e., the space of the antilinear functionals in the space Φ. The triplet of the spaces Φ, $\mathcal{H}$, and $\Phi^\times$ fulfills the relation

$$\Phi \subset \mathcal{H} \subset \Phi^\times \tag{6}$$

and is called the rigged Hilbert space or the Gel'fand triplet.

Let us now consider the self-adjoint operator A in the space Φ. Then we define the operator $A^\times$ which is the extension of the operator A by

$$\langle \varphi | A^\times F \rangle = \langle A\varphi | F \rangle \quad \text{for all} \quad \varphi \in \Phi (F \in \Phi^\times). \tag{7}$$

If the operator $\overline{A}$ is the closure of A in $\mathcal{H}$ then we have the following triplet of the operators

$$A \subset \overline{A} \subset A^\times \tag{8}$$

which are defined in their corresponding spaces.

Now we will introduce the following:

Definition 2.1. $F \in \Phi^\times$ is the *generalized eigenvector* of the operator $A^\times$ if the following relation holds

$$\langle \varphi | A^\times F \rangle = \langle A\varphi | F \rangle = \lambda \langle \varphi | F \rangle \quad \text{for all} \quad \varphi \in \Phi \tag{9}$$

and λ is the generalized eigenvalue. Relation (9) is also written in the following equivalent form

$$A^{\times} F = \lambda F. \tag{10}$$

The notion of the generalized eigenvector allows to introduce the nuclear spectral theorem (Gel'fand and Maurin (Gel'fand and Vilenkin, 1964; Maurin, 1968)).

Theorem 1. *(Nuclear Spectral Theorem). Let $\Phi \subset \mathcal{H} \subset \Phi^{\times}$ be a rigged Hilbert space and A be a self-adjoint, cyclic, operator continuous in Φ. Then there exists a set of the generalized eigenvectors F_{λ} such that for every $\varphi, \psi \in \Phi$ the following relation holds*

$$(\varphi|\psi) = \int_{\Lambda} d\lambda \langle \varphi | F_{\lambda} \rangle \langle F_{\lambda} | \psi \rangle \tag{11}$$

where Λ is the spectrum of the operator $\overline{A}$. The nuclear spectral theorem states that the set of the generalized eigenvectors forms the complete set of the vectors. The classical example of this formalism is the Dirac formalism of bras and kets where the generalized eigenvectors of the position and momentum operators fulfill

$$I = \int dx |x\rangle\langle x| = \int dp |p\rangle\langle P| \tag{12}$$

2.2. Rigged Hilbert Spaces in the Scattering Experiment

Let us now discuss a scattering experiment. Conventionally it is divided into three phases

1. Preparation
2. Interaction
3. Observation

The space of the prepared states and the space of the observables are considered equal and are identified with the Hilbert space of the physical system. The absence of the resonance states from Hilbert space means that the conventional quantum mechanics cannot precisely answer various questions about the intermediate states and the dynamics of the scattering process.

The plane wave scattering states fulfill the Lippmann–Schwinger equation

$$|E^{\pm}\rangle = |E\rangle + \frac{1}{E - H_0 \pm i\varepsilon} V |E^{\pm}\rangle \tag{13}$$

where $|E^{\pm}\rangle$ are the eigenstates of the full Hamiltonian $H = H_0 + V$ and $|E\rangle$ are the eigenstates of the free Hamiltonian H_0 and the $\pm$ superscript corresponds to the in- and out-going plane waves. The states $|E^{\pm}\rangle$ in conventional quantum mechanics are poorly defined since the $|E^{\pm}\rangle$ and the $|E\rangle$ are generalized vectors and do not belong to the Hilbert space. They are elements of the space $\Phi^{\times}$. The mathematical properties of the states $|E^{\pm}\rangle$ will be obtained when we consider an element of the S matrix for a scattering experiment with an intermediate resonance.

Let us consider a scattering experiment with an intermediate unstable state. The partial wave S matrix for such a process has a resonance pole on the unphysical energy sheet

$$S_j(\mathrm{s}) = \frac{R}{\mathrm{s} - \mathrm{s}_R} + R_0 + R_1(\mathrm{s} - \mathrm{s}_R) + \cdots \tag{14}$$

where s_R is the position of the resonance pole

$$\mathrm{s}_R = \left(E_R - \frac{i}{2}\Gamma_R\right)^2 \tag{15}$$

and $\mathrm{s} = p^2$ is the Mandelstam variable representing the square of the center-of-mass energy of the process.[3]

Let us consider now the S matrix element of the scattering process where ϕ and ψ correspond to the prepared and detected states

$$\begin{aligned}(\psi^{\mathrm{out}}|\phi^{\mathrm{out}}) &= (\psi^{\mathrm{out}}|S\phi^{\mathrm{in}}) = (\Omega^{-}\psi^{\mathrm{out}}|\Omega^{+}\phi^{\mathrm{in}}) \\ &= (\psi^{-}|\phi^{+}) = \sum_{j,j_3,n}\int \frac{d_3\hat{\mathbf{p}}}{2\hat{E}}\,ds \sum_{j',j_3',n'} \frac{d_3\hat{\mathbf{p}}'}{2\hat{E}'}\,ds'\langle\psi^{-}|[js]nj_3\hat{\mathbf{p}}^{-}\rangle \\ &\quad \times \langle\hat{\mathbf{p}}j_3n[js]|S|[j'\mathrm{s}']n'j_3'\hat{\mathbf{p}}'\rangle\langle^{+}\hat{\mathbf{p}}'j_3'n'[j'\mathrm{s}']|\phi^{+}\rangle\end{aligned} \tag{16}$$

Here $\Omega^{\pm}$ are the Møller operators and the states $|[js]nj_3\hat{\mathbf{p}}^{\pm}$ are the solutions of the Lippmann–Schwinger equation and are the generalized eigenvectors of the exact energy operator $P^{\mu}P_{\mu}$ with the eigenvalue $p^2 = s$ and the total angular momentum j. In Eq. (16) we have inserted twice the completeness relation following from the nuclear spectral the-orem (11) for the scattering states. Notice that for the reasons explained later we are using the 4-velocity eigenkets, this means that $\hat{\mathbf{p}}$ in (16) is given by $\hat{\mathbf{p}} = \mathbf{p}/\sqrt{\mathrm{s}}$. This procedure is compatible with the Poincaré invariance and does not violate any principles. The Poincaré invariance for the S matrix contains the energy–momentum conservation which can be expressed as

$$\langle\hat{\mathbf{p}}j_3n[js]|S|[j'\mathrm{s}']n'j_3'\hat{\mathbf{p}}'\rangle = 2\hat{E}\delta_3(\hat{\mathbf{p}} - \hat{\mathbf{p}}')\delta(\mathrm{s} - \mathrm{s}')\delta_{j,j'}\delta_{j_3,j_3'}S_j(\mathrm{s}) \tag{17}$$

where $S_j(\mathrm{s})$ is the reduced matrix element.

Now using Eq. (17) we can write the j-th partial wave matrix element of Eq. (16) in the following symbolic form

$$(\psi^{\mathrm{out}}|\phi^{\mathrm{out}})_j = (\psi^{-}|\phi^{+})_j = \int_{m_0^2}^{\infty} ds\langle^{-}\psi|s^{-}\rangle S_j(\mathrm{s})\langle^{+}\mathrm{s}|\phi^{+}\rangle \tag{18}$$

[3] In the process $1 + 2 \to R \to 3 + 4 \mathrm{s} = (p_3 + p_4)_{\mu}(p^3 + p^4)^{\mu}$ and the threshold of the process is $m_0 = m_3 + m_4$.

where m_0^2 is the threshold of the process and where we use the abbreviation $\langle^+ s|\phi^+\rangle = \langle^+\hat{\mathbf{p}}\, j_3 n[js]|\phi^+\rangle$ and similarly for $\langle^- \mathrm{s}|\psi^-\rangle = \overline{\langle^-\psi|\mathrm{s}^-\rangle}$. In Eq. (18) we have suppressed the integration over the velocities $\hat{\mathbf{p}}$, which is implied.

Now we would wish to use the analyticity property that follows from the presence of the resonance, Eq. (14). The s integration in Eq. (18) is on the physical sheet and the pole in Eq. (14) is on the unphysical sheet. To be able to use Eq. (14) we must modify the contour of the integration and for this reason we must know the analyticity properties of the full integrand in Eq. (18), i.e., also of $\langle^-\psi|\mathrm{s}^-\rangle$ and $\langle^+\mathrm{s}|\phi^+$. Now, the minimal analyticity assumption that allows the contour modification to the second sheet is that $\langle^-\psi|\mathrm{s}^-$ and $\langle^+\mathrm{s}|\phi^+$ are analytical functions of s in the lower half plane and that they do not grow too fast for large values of s. Mathematically it is expressed that $\langle^-\mathrm{s}|\psi^-\rangle$ and $\langle^+\mathrm{s}|\phi^+\rangle$ are smooth Hardy class functions from above and below, respectively

$$\begin{aligned}\langle^-\hat{\mathbf{p}}\, j_3[js]|\psi^-\rangle &\equiv \langle^-\mathrm{s}|\psi^-\rangle \in \mathcal{H}^2_+ \cap \mathcal{S} = \Phi_+\\ \langle^+\hat{\mathbf{p}}\, j_3[js]|\phi^+\rangle &\equiv \langle^+\mathrm{s}|\phi^+\rangle \in \mathcal{H}^2_- \cap \mathcal{S} = \Phi_-.\end{aligned} \tag{19}$$

Here $\mathcal{H}^2_\pm$ denotes the space of Hardy class functions from above and below and $\mathcal{S}$ is the Schwartz space.[4]

Equation (19) allows the contour modification shown in Fig. 1 and Eq. (18) then reads

$$\begin{aligned}(\psi^-|\phi^+)_j &= \int_{m_0^2}^{\infty} d\mathrm{s}\langle^-\psi|\mathrm{s}^-\rangle S_j(\mathrm{s})\langle^+\mathrm{s}|\phi^+\rangle\\ &= \int_{m_0^2}^{-\infty II} d\mathrm{s}\langle^-\psi|\mathrm{s}^-\rangle S_j^{\mathrm{II}}(\mathrm{s})\langle^+\mathrm{s}|\phi^+\rangle\\ &\quad + \oint_{C_\mathrm{R}} d\mathrm{s}\langle^-\psi|\mathrm{s}^-\rangle S_j^{\mathrm{II}}(\mathrm{s})\langle^+\mathrm{s}|\phi^+\rangle.\end{aligned} \tag{20}$$

The first integral in the range $[m_0^2, -\infty_{\mathrm{II}}]$ is known as the background integral and it is the slowly varying, regular function of s (the integration path is far from the resonance pole). The second integral, along C_R is around the resonance pole

$$\begin{aligned}&(\psi^-|\phi^+)_{j,\text{pole term}} = \oint_{C_\mathrm{R}} ds\langle^-\psi|\mathrm{s}^-\rangle S_j^{\mathrm{II}}(\mathrm{s})\langle^+\mathrm{s}|\phi^+\rangle\\ &= -2\pi i R_{-1}\langle^-\psi|\mathrm{s}_\mathrm{R}^-\rangle\langle^+\mathrm{s}_\mathrm{R}|\phi^+\rangle = \int_{\infty\mathrm{II}}^{\infty} d\mathrm{s}\langle^-\psi|\mathrm{s}^-\rangle\langle^+\mathrm{s}|\phi^+\rangle\frac{R_{-1}}{\mathrm{s}-\mathrm{s}_\mathrm{R}}.\end{aligned} \tag{21}$$

[4] Note the incompatibility of the signs of the notation for the in- and out-going states and for the Hardy spaces, which follows the traditional notation in physics for the states and in mathematics for the Hardy spaces.

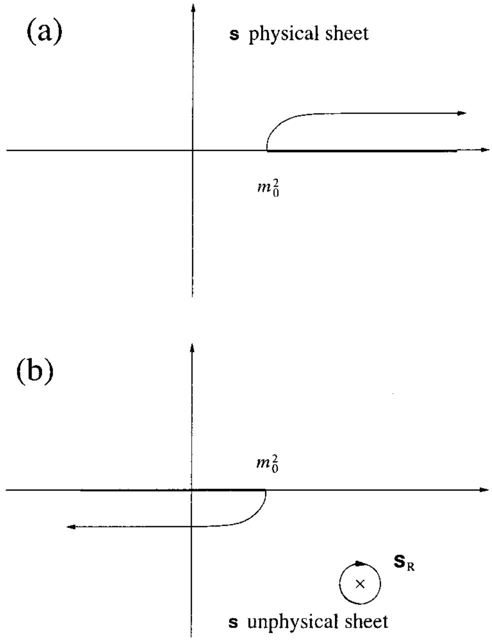

Fig. 1. Integration contours in Eq. (18). (a) Integration contour before the deformation. (b) Modified contour on the unphysical s sheet includes the integration values of s below m_0^2 and the integration around the pole in the position s_R

Equation (21) holds for any vector $|\psi^-\rangle \in \Phi_+$, i.e. we can omit this vector from Eq. (21) and this leads to the definition of the vector for the unstable state (Gamow vector)

$$|[j s_R] b^-\rangle = \frac{i}{2\pi} \int_{-\infty}^{\infty} ds |[js]b^-\rangle \frac{1}{s - s_R}. \tag{22}$$

where b denotes the remaining quantum numbers including the velocity $\hat{\mathbf{p}}$.

We have indicated earlier that we were using the velocity eigenstates instead of the momentum eigenstates. The reason for this, though technical, is important. In the derivation of Eq. (20) we were modifying the integration path for the variable $s = \sqrt{\mathbf{p}^2 + m^2}$. Going to complex s means going to complex momentum $\mathbf{p}$. If one uses velocity $\hat{\mathbf{p}} = \mathbf{p}/m$, one can choose to keep the 4-velocity $\hat{\mathbf{p}}$ real while one goes to complex s. Such a choice is necessary because we thus avoid the use of the Lorentz group representations in which boost will depend upon complex parameters. These representations with complex momentum but real velocity are called minimally complex.

Another important point is the property that the in- and out-states form two rigged Hilbert spaces based on the spaces of the Hardy class functions from above and below. The distinction of these spaces was needed to obtain from the intermediate state the specific properties of the Gamow vectors (Bohm, 1978b, 1981). This assumption of two Hardy spaces will also turn out to be the source of the irreversible evolution that will be discussed in the next section.

3. PHYSICAL PROPERTIES OF THE RELATIVISTIC GAMOW STATES

No further new assumptions than the Hardy space assumption are necessary to establish the properties of the states (22). The most important of these are

1. The states (22) are the generalized eigenstates of the operators

$$P^\mu P_\mu,\ P^0 = H = H_0 + V$$

e.g.,

$$P^\mu P_\mu |[j\mathrm{s}_\mathrm{R}]b^-\rangle = \mathrm{s}_\mathrm{R}|[j\mathrm{s}_\mathrm{R}]b^-\rangle. \tag{23}$$

2. The time evolution of the ket $|[j\mathrm{s}_\mathrm{R}]b^-\rangle$ is allowed only for $t \geq 0$ and the following relation holds

$$e^{-H^\times t}|[j\mathrm{s}_\mathrm{R}]b^-_{\mathrm{rest}}\rangle = e^{-i\sqrt{\mathrm{s}_\mathrm{R}}t}|[j\mathrm{s}_\mathrm{R}]b^-_{\mathrm{rest}}\rangle, \quad \text{for} \quad t \geq 0 \text{ only.} \tag{24}$$

The proof of Eq. (23) follows from the fact that if $\psi^- \in \Phi_+ \Rightarrow P^\mu P_\mu \psi^- \in \Phi_+$ and

$$\begin{aligned}\langle\psi^-|(P^\mu P_\mu)^\times[j\mathrm{s}_\mathrm{R}]b^-\rangle &= \langle P^\mu P_\mu\psi^-|[j\mathrm{s}_\mathrm{R}]b^-\rangle \\ &= \frac{i}{2\pi}\int_{\infty\mathrm{II}}^{+\infty} d\mathrm{s}\langle P^\mu P_\mu\psi^-|[j\mathrm{s}]b^-\rangle\frac{1}{\mathrm{s}-\mathrm{s}_\mathrm{R}} \\ &= \frac{i}{2\pi}\int_{\infty\mathrm{II}}^{+\infty} d\mathrm{s}\langle\psi^-|(P^\mu P_\mu)^\times[j\mathrm{s}]b^-\rangle\frac{1}{\mathrm{s}-\mathrm{s}_\mathrm{R}} \\ &= \frac{i}{2\pi}\int_{\infty\mathrm{II}}^{+\infty} d\mathrm{s}\langle\psi^-|[j\mathrm{s}]b^-\rangle\mathrm{s}\frac{1}{\mathrm{s}-\mathrm{s}_\mathrm{R}} \\ &= \mathrm{s}_\mathrm{R}\langle\psi^-|[j\mathrm{s}_\mathrm{R}]b^-\rangle \text{ for all } \psi^- \in \Phi_+,\end{aligned} \tag{25}$$

which is equivalent to

$$(P^\mu P_\mu)^\times|[j\mathrm{s}_\mathrm{R}]b^- = \mathrm{s}_\mathrm{R}|[j\mathrm{s}_\mathrm{R}]b^-\rangle. \tag{26}$$

The eigenvalue of H on the rest states is given by

$$(H)^\times|[j\mathrm{s}_\mathrm{R}]b^-_{\mathrm{rest}}\rangle = \sqrt{\mathrm{s}_\mathrm{R}}|[j\mathrm{s}_\mathrm{R}]b^-_{\mathrm{rest}}\rangle. \tag{27}$$

In Eq. (24) the most interesting and important property is that it holds only for $t \geq 0$. This fact is a consequence of the Hardy space assumption (19) $\psi^- \in \Phi^+$ for out-observables ψ^-. If with $\psi^- \in \Phi^+$ also the transformed state $\exp(iHt)\psi^-$ should belong to Φ_+ and this can only hold for $t \geq 0$. From (24) we see that the time evolution of the Gamow state at rest $|[j\mathsf{s}_{\mathrm{R}}]b^-_{\mathrm{rest}}\rangle$ is the following

$$e^{-iH^\times t}|[j\mathsf{s}_{\mathrm{R}}]b^-_{\mathrm{rest}}\rangle = e^{-iE_{\mathrm{R}}t}\, e^{-\Gamma_{\mathrm{R}}t/2}|[j\mathsf{s}_{\mathrm{R}}]b^-_{\mathrm{rest}}\rangle \quad \text{for} \quad t \geq 0. \tag{28}$$

Again this result is the consequence of the Hardy space assumption (19) which constitutes asymmetric boundary conditions.

The transformation properties of the Gamow kets under the Lorentz transformation can also be established. If $\mathcal{U}(\Lambda)$ is the operator of the Lorentz transformation Λ in the Hilbert space $\mathcal{H}$ then we have

$$(\mathcal{U}(\Lambda))^\times|[j\mathsf{s}_{\mathrm{R}}]\hat{\mathbf{p}}\,j_3^- = \sum_{j_3'} |[j\mathsf{s}_{\mathrm{R}}]\Lambda^{-1}\hat{\mathbf{p}}\,j_3'^-\rangle D^j_{j_3' j_3}(\mathcal{R}(\Lambda^{-1}, \hat{\mathbf{p}})), \tag{29}$$

where $\mathcal{R}(\Lambda, \hat{p}) = L^{-1}(\Lambda, \hat{\mathbf{p}})\Lambda L(\hat{\mathbf{p}})$ is the Wigner rotation and $D^j_{j_3' j_3}(\mathcal{R})$ are the matrix elements of the irreducible representation of the rotation group with the angular momentum j. The transformation law of the states $|[j\mathsf{s}_{\mathrm{R}}]\hat{\mathbf{p}}\,j_3^-$ under the Lorentz boosts $\Lambda = L(\hat{\mathbf{p}})$ is the demonstration that these states form an irreducible representation of the homogeneous Lorentz group.

As discussed earlier the space Φ_+ is not invariant under the action of the time translation $\exp(-iHt)$. It is invariant only under the semigroup of the time translations for $t \geq 0$. These results can be summarized as

> Gamow kets form the semigroup representation of the Poincaré group with spin j and complex invariant mass square $\mathsf{s}_{\mathrm{R}} = (E_{\mathrm{R}} - \frac{i}{2}\Gamma_{\mathrm{R}})^2$ (30)

This means that from the theoretical point of view there exists the unified description of the stable states and resonances, stable states are in a certain sense the limiting case $\Gamma_{\mathrm{R}} \to 0$.

The probabilities to find in a state ϕ the observable $|\psi\rangle\langle\psi|$ are in quantum theory given by the Born probabilities

$$\mathcal{P}_\psi(\phi(t)) = \mathrm{Tr}(|\phi(t)\rangle\langle\phi(t)|\psi\rangle\langle\psi|) = |\langle e^{iHt}\psi|\phi\rangle|^2 = |\langle\psi|e^{-iHt}\phi\rangle|^2. \tag{31}$$

This interpretation can be extended to the elements of $\Phi_+^\times$ like the Gamow state $|[j\mathsf{s}_{\mathrm{R}}]\hat{\mathbf{p}}\,j_3^-$. The probability per unit time to register by the detector ΔN counts of the decay products described by the observable $|\psi^-\rangle\langle\psi^-|$ is—as a generalization of (31)—proportional to the absolute value square of the amplitude

$$\begin{aligned}&\langle e^{iHt}\psi^-|[j\mathsf{s}_{\mathrm{R}}]\hat{\mathbf{p}} = 0\, j_3\rangle \\ &\qquad = e^{-iE_{\mathrm{R}}t}\, e^{-\Gamma_{\mathrm{R}}t/2}\langle\psi^-|[j\mathsf{s}_{\mathrm{R}}]\hat{\mathbf{p}} = 0\, j_3\rangle \quad \text{for } t \geq \text{only}.\end{aligned} \tag{32}$$

The conclusions from this mathematical consequence (32) are

1. The Gamow vector with Breit–Wigner resonance width Γ_R defined by Eq. (15) from the S-matrix pole at s_R decays exponentially in time since the counting rate $\Delta N(t)/\Delta t$ is proportional to

$$|\langle\psi^-|e^{-iH^\times t}|[j s_R]\hat{\mathbf{p}} = 0 j_3\rangle^2 = e^{-\Gamma_R t}|\langle\psi^-|[j s_R]\hat{\mathbf{p}} = 0 j_3\rangle|^2. \tag{33}$$

   Because of the exponential time dependence in Eq. (33) the lifetime τ of the Gamow state is given by $\tau = 1/\Gamma_R$.
2. The time evolution of the Gamow vectors (at rest) is time asymmetric $t \geq 0$ given by the semigroup $U^\times(t) = e^{-iH^\times}t$. This quantum mechanical irreversibility on the microphysical level appears at first shocking but it is consistent with the principle of causality and means that the Gamow state must be prepared (at $t = t_0 = 0$) before an observable $|\psi^-\rangle\langle\psi^-$ can be detected in it at $t > t_0$ (Bohm *et al.*, 1997). The Hilbert space unitary evolution $U(t) = e^{-iHt}$, $-\infty < t < \infty$ of the observable $|\psi\rangle\langle\psi|$ permits their measurement for $t \leq 0$. i.e., before the state was prepared, which is in conflict with causality.

4. THE PROBLEM OF THE *Z* BOSON MASS

4.1. Phenomenology of Resonances

Resonances and quasistable particles are characterized by two sets of real numbers (E_R, Γ), or (E_R, $R = \frac{1}{\tau}$), respectively. Lifetime τ and its inverse $R \equiv \frac{1}{\tau}$, the initial decay rate, are measured by fits of the counting rate of the decay products $\Delta N/\Delta t$ to the exponential law $\exp(-t/\tau)$ whereas the width Γ is measured by fits of the cross-section to the Lorentzian (Breit–Wigner) energy distribution (η denotes the decay channel)

$$\sigma_j^{\mathrm{BW}}(E) \sim |a_j^{\mathrm{BW}}(E)|^2 = \left|\frac{r_\eta}{E - \left(E_R - i\frac{\Gamma}{2}\right)}\right|^2$$
$$\sim \frac{1}{(E - E_R)^2 - \left(\frac{\Gamma}{2}\right)^2}, \quad \text{with} \quad 0 \leq E < \infty. \tag{34}$$

(plus usually some background term $B(E)$). The initial decay rate $R(0) = R \equiv 1/\tau$ and the resonance width Γ are thus different quantities; the decay rate

$$R(t) = \sum_\eta R_\eta(t) = \sum_\eta R_\eta(0)e^{-\frac{t}{\tau}} = Re^{-\frac{t}{\tau}} \tag{35}$$

is connected with the exponential time evolution and Γ is connected with the Lorentzian energy distribution (34). However, it became common practice not to distinguish between the rate R and the width Γ and to identify $\frac{\hbar}{\tau} \equiv R$ with Γ

$$\Gamma = \frac{\hbar}{\tau} \equiv R \tag{36}$$

In nonrelativistic physics this relation is justified by the Wigner–Weisskopf approximation. In the nonrelativistic theory based on the rigged Hilbert space approach using the Hardy space assumption (19). The formula (36) can also be proven as an exact result for the (nonrelativistic) Gamow vector (Bohm, 1999). Here we will discuss the problem of resonances in relativistic physics where the common opinion is quite different. Resonances are defined by the perturbation theoretical definition using the self-energy of the propagators and are considered as complicated objects that cannot be described as an exponentially decaying state or as a state characterized by two numbers like (E, Γ).

The j-th partial scattering amplitude in a relativistic resonance for-mation process $a_j(\mathrm{s})$ is a function of invariant mass square $\mathrm{s} = (p_1^\mu p_2^\mu)^2 = (E_1^{\mathrm{cm}} + E_2^{\mathrm{cm}})^2$, where p_1^μ, p_2^μ are the momenta of the two incoming (or outgoing) particles. One writes the amplitude of a resonance scattering process as

$$a_j(s) = a_j^{\mathrm{res}}(\mathrm{s}) + B_j(s) \tag{37}$$

where $B_j(\mathrm{s})$ is the nonresonant (constant) background and a_j^{res} is the contribution of the resonance for which one uses various assumptions all called relativistic Breit–Wigner energy distributions. However, in contrast to the nonrelativistic case, where one defines resonance energy and resonance width by one Breit–Wigner formula (34) and has one definition of $(E_\mathrm{R}, \Gamma_\mathrm{R})$ in the relativistic case one does not have a universally agreed upon definition of the resonance amplitude and of resonance mass and width.

4.2. Standard Model

The standard model of the elementary particles is formulated in terms of the field theory with the gauge symmetry based on the group $SU(2) \times U(1) \times SU(3)$. The subgroup $SU(2) \times U(1)$ of the gauge group describes the unified electro–weak interactions and the subgroup $SU(3)$ is responsible for strong interactions (quantum chromodynamics). For each subgroup of the gauge group there is one independent coupling constant so in the standard model there are three independent coupling constants. The elementary "material" particles of spin $1/2$ are quarks and leptons and there are three generations of each. Additionally each quark appears in three colors while leptons are colorless. This is the reason why quarks interact strongly and leptons do not.

The interactions in the standard model are determined from the gauge invariance which requires that for each generator of the gauge group there is one spin and one boson. It means that in quantum chromodynamics there are eight electrically neutral bosons called gluons (the group $SU(3)$ has eight generators) and for the electro–weak interactions there are four bosons: two charged and two neutral (the group $SU(2) \times U(1)$ has four generators). The gauge invariance requires that all particles of the theory are massless what is in clear contradiction with the reality. The masses of the "material" particles and of the gauge bosons can be generated without spoling the renormalizability of the theory by the spontaneous symmetry breaking and the Higgs mechanism (which requires additional spin 0 particles) with the photon and gluons remaining massless. Until now all the particles with the exception of the Higgs boson have been experimentally observed either directly or indirectly (gluons).

Let us now focus our attention on the electro–weak vector (spin 1) bosons. The charged vector bosons describe the weak interactions with the transfer of the electric charge (e.g. in the decay of the neutron into the proton, electron and antineutrino $n \to p + e + \overline{\nu}$, the initial neutron charge is 0 and the final proton has charge 1, i.e. the neutron emits the W^- boson and the proton $n \to p + W^-$ and W^- subsequently decays into an electron and neutrino, $W^- \to e + \overline{\nu}$. Before the unification of the weak and electromagnetic interactions it was believed that there are no neutral current weak interactions. Only after the formulation of the unified electro–weak theory, which requires such interactions for the theoretical consistency, such interactions were discovered. These interactions are mediated by the neutral massive Z boson ("heavy photon"). In summary: the electro–weak interactions are mediated by the four vector bosons W^+, W^-, Z, and γ. The photon γ is massless and the masses of the $W^\pm$ and Z bosons were predicted prior to their discovery to be $\sim$80 GeV and $\sim$90 GeV (80 and 90 times heavier than the proton), respectively. The experimental discovery of the bosons $W^\pm$ and Z with the properties exactly as predicted was a very strong confirmation of the unified model of the electro–weak interactions.

4.3. Detailed Analysis of the Z-Boson Mass

The mass and width of the Z boson have been measured with an extraordinary precision and the Review of Particle Properties (Hagiwara *et al.*, 2002) gives two definitions of the mass and width of the Z-boson and lists two different values which are obtained from the fit of two different formulas for the lineshape to the same experimental data. The value M_Z is obtained from the fit to the "relativistic Breit–Wigner with energy dependent width" of the on-shell renormalization scheme (Bernicha *et al.*, 1994, 1996; Hagiwara *et al.*, 2002; Sirlin, 1991a, b; Stuart, 1991, 1997; Willenbrock and Valencia, 1991)

$$\begin{aligned} a_j^{\mathrm{om}}(\mathrm{s}) &= \frac{-\sqrt{\mathrm{s}}\sqrt{\Gamma_e(\mathrm{s})\Gamma_f(\mathrm{s})}}{\mathrm{s} - M_Z^2 + i\sqrt{\mathrm{s}}\Gamma_Z(\mathrm{s})} \approx \frac{-M_Z B_{ef}\Gamma_Z}{\mathrm{s} - M_Z^2 + i\frac{\mathrm{s}}{M_Z}\Gamma_Z} \\ &= \frac{R_Z}{\mathrm{s} - M_z^2 + i\frac{\mathrm{s}}{M_Z}\Gamma_Z}, \quad m_0^2 \le \mathrm{s} < \infty. \end{aligned} \tag{38}$$

The value $\bar{M}_Z$ is obtained from the relativistic Breit–Wigner of the S-matrix pole

$$a_j^{\mathrm{BW}}(\mathrm{s}) = \frac{R_z}{\mathrm{s} - \mathrm{s}_R} = \frac{R_z}{\mathrm{s} - \bar{M}_Z^2 + i\bar{M}_Z\bar{\Gamma}_Z}$$
$$= \frac{R_z}{\mathrm{s} - \left(M_R - i\frac{\Gamma_R}{2}\right)^2}, \quad m_0^2 \leq \mathrm{s} < \infty. \tag{39}$$

Both lineshape formulas (38) and (39) reproduce the experimental data equally well. But they lead to values of the mass parameters M_Z and $\bar{M}_Z$ which differ from each other by about 10 times the experimental error. From $a_j^{\mathrm{om}}(\mathrm{s})$ the fit gives the values

$$M_z = (91.1871 \pm 0.0021)\ \mathrm{GeV}, \qquad \Gamma_Z = (2.4945 \pm 0.0024)\ \mathrm{GeV}. \tag{40}$$

From $a_j^{\mathrm{BW}}(\mathrm{s})$ one obtains the values $\bar{M}_Z$, $\bar{\Gamma}_Z$ and M_R, Γ_R and these parameters numerically differ from each other by

$$M_R = M_Z - 0.026\ \mathrm{GeV} = M_z - 10 \times \Delta M_Z, \qquad \Gamma_R = \Gamma_Z - 1.2\ \mathrm{MeV}, \tag{41}$$

and

$$\bar{M}_Z = M_Z - 0.0341\ \mathrm{GeV}, \qquad \bar{\Gamma}_Z = \Gamma_Z - 1.2\ \mathrm{MeV} \tag{42}$$

The question thus is: what is the right definition of the Z-boson mass and width and therefore the right numerical value of the mass of the Z-boson?

Even if one discards the on-shell definition (38) and chooses the S-matrix definition (39) because it is gauge invariant, the complex parameters s_R in Eq. (39) can be expressed in terms of the real parameters mass and width in many different ways leading to many arbitrary definitions of the Z-boson mass. Some of these mentioned in the literature of which (40), (41), (42) are just a few.

Our answer to the question what is the right definition and therefore what is the correct mass value obtained from the data is given by the Poincaré transformations. The states of stable elementary particles have been defined since Wigner (1939; Bargmann and Wigner, 1948) as vectors of an irreducible unitary representation space $[jm^2]$ of the Poincaré group $\mathcal{P}$. This should not be restricted to interaction free, asymptotic states, but apply also to the exact states and to Poincaré transformations generated by the (interaction-incorporating) *exact generators* $P_0 = H = H_0 + V$, P^i, $J^{\mu\nu}$. This means we use the relativistic Gamow vectors (22) with relativistic energy distribution given by the relativistic Breit–Wigner formula (39). From Eq. (32) for the time dependence of the decay rate = counting rate we conclude that the lifetime of the Gamow state and the width of the resonance state are related by

$$\tau = \frac{1}{\Gamma_{\mathrm{R}}}, \tag{43}$$

where Γ_R is related to the S matrix pole position s_R by

$$s_R = \left(M_R - i\frac{\Gamma_R}{2} \right)^2 \tag{44}$$

and M_R and Γ_R label the irreducible representation of the Poincaré semigroup of the Gamow states. The Poincaré invariance together with the relation (43) uniquely determines the mass and width of an unstable state.

5. CONCLUSIONS

We have reviewed here the rigged Hilbert space extension of quantum mechanics. It should be pointed out that for the stationary states of stable objects the rigged Hilbert space quantum mechanics reduces to the conventional results. It means that the rigged Hilbert space quantum mechanics does not contradict in any way the ordinary quantum mechanics. The three main advantages of the rigged Hilbert space quantum mechanics are the following. First, as was clear from the discussion of the Dirac bras and kets, it adds the mathematical precision and better understanding to some sectors of the ordinary quantum mechanics. Second, and more importantly, it extends quantum mechanics to a theory which includes unstable states and thus gives the possibility to discuss these physical phenomena in a mathematically precise way. Finally, it introduces with the Hardy space assumption (19) a new quantum mechanical arrow of time which distinguishes the semigroup representations of the Poincaré transformations into the forward light cone.

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